Quasi-morphisms and the Poisson bracket

Michael Entov[a], Leonid Polterovich[b] and Frol Zapolsky[b]

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To Gregory Margulis with admiration

Abstract

For a class of symplectic manifolds, we introduce a functional which assigns a real number to any pair of continuous functions on the manifold. This functional has a number of interesting properties. On the one hand, it is Lipschitz with respect to the uniform norm. On the other hand, it serves as a measure of non-commutativity of functions in the sense of the Poisson bracket, the operation which involves first derivatives of the functions. Furthermore, the same functional gives rise to a non-trivial lower bound for the error of the simultaneous measurement of a pair of non-commuting Hamiltonians. These results manifest a link between the algebraic structure of the group of Hamiltonian diffeomorphisms and the function theory on a symplectic manifold. The above-mentioned functional comes from a special homogeneous quasi-morphism on the universal cover of the group, which is rooted in the Floer theory.

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1 Introduction and main results

1.1 Symplectic quasi-states

Let (M^{2n}, ω) be a closed connected symplectic manifold. A *symplectic quasi-state* is a functional $\zeta : C^0(M) \to \mathbb{R}$ with the following properties:

(i) $\zeta(1) = 1$;

(ii) $F \leq G \Rightarrow \zeta(F) \leq \zeta(G)$;

(iii) $\zeta(aF + bG) = a\zeta(F) + b\zeta(G)$ for all $a, b \in \mathbb{R}$ and all functions $F, G \in C^\infty(M)$ whose Poisson bracket $\{F, G\}$ vanishes.

In particular, ζ is a topological quasi-state in the sense of Aarnes [1]. It was shown in [6] that certain symplectic manifolds carry a **non-linear** symplectic quasi-state ζ. In this case the functional

$$\Pi : C^0(M) \times C^0(M) \to \mathbb{R}$$

defined by

$$\Pi(F, G) = |\zeta(F + G) - \zeta(F) - \zeta(G)| \tag{1}$$

can be interpreted as a measure of Poisson non-commutativity of functions F and G. This functional lies in the center of our study. In particular, we analyze the relation between $\Pi(F, G)$ and the Poisson bracket $\{F, G\}$. We show (see Theorem 1.4 below) that for certain symplectic quasi-states

$$\Pi(F, G) \leq \text{const} \cdot \sqrt{||\{F, G\}||} \tag{2}$$

for all $F, G \in C^\infty(M)$. Here and below $||H||$ stands for the uniform norm $\max_M |H|$. This inequality has a number of applications.

One application deals with the following problem (cf. [4]). The definition of the Poisson bracket $\{F, G\}$ of two functions F, G on a symplectic manifold M involves first derivatives of the functions. Thus *a priori* there is no restriction on possible changes of $\{F, G\}$ when F and G are perturbed in the uniform norm. Note that axiom (ii) of the quasi-state yields that Π is Lipschitz in the uniform norm. Therefore inequality (2) gives rise to such a restriction (see Corollary 1.5 below).

As another application, we present a restriction on partitions of unity $\rho_1, \ldots, \rho_N$ on symplectic manifolds subordinate to coverings by sufficiently small sets (see Theorems 1.7 and 1.8 below). It turns out that

$$\max_{i,j} ||\{\rho_i, \rho_j\}|| \geq \frac{\text{const}}{N^3} \ .$$

Interestingly enough, the functional Π appears in the context of simultaneous measurements of non-commuting observables F, G in classical mechanics (see Section 1.6 below). We show that $\Pi(F,G)$ gives a lower bound for the error of such a measurement.

The above-mentioned quasi-states are closely related to certain homogeneous quasi-morphisms (that is, homomorphisms up to a bounded error) on the universal cover $\mathcal{G}$ of the group of Hamiltonian diffeomorphisms of M. These quasi-morphisms, which were found in [5], are rooted in Floer homology. The connection between quasi-states and quasi-morphisms is crucial for our methods.

1.2 Preliminaries on Hamiltonian diffeomorphisms

In what follows we normalize the symplectic form ω on M^{2n} so that the symplectic volume $\int_M \omega^n$ equals 1. Recall that symplectic manifolds appear as phase spaces in classical mechanics. An important principle of classical mechanics is that *the energy of a system determines its evolution.* The energy (or *Hamiltonian function*) $F_t(x) := F(x,t)$ is a smooth function on $M \times [0;1]$. Here t is the time coordinate. Define the time-dependent Hamiltonian vector field $\operatorname{sgrad} F_t$ by the point-wise linear equation $i_{\operatorname{sgrad} F_t}\omega = -dF_t$. The evolution of the system is described by the flow f_t on M generated by the Hamiltonian vector field $\operatorname{sgrad} F_t$. We shall call the time-one-map f_1 of this flow *a Hamiltonian diffeomorphism.* Hamiltonian diffeomorphisms form a group which is denoted by $Ham(M,\omega)$. The universal cover $\mathcal{G}$ of this group plays an important role in this paper. Elements of $\mathcal{G}$ are smooth paths in $Ham(M,\omega)$ based at the identity, considered up to homotopy with fixed end-points. We denote by ϕ_F the element of $\mathcal{G}$ represented by the path $\{f_t\}_{t\in[0;1]}$ and refer to it as the element of $\mathcal{G}$ generated by F.

It is instructive to view $\mathcal{G}$ as a (infinite-dimensional) Lie group whose Lie algebra is naturally identified with the space $C_0^\infty(M)$ of smooth functions with zero mean on M. The role of the Lie bracket is played by the Poisson

bracket which is defined by

$$\{F, G\} = \omega(\mathrm{sgrad}\, G, \mathrm{sgrad}\, F) = dF(\mathrm{sgrad}\, G) = -dG(\mathrm{sgrad}\, F)\ .$$

In the canonical local coordinates (p, q) where $\omega = dp \wedge dq$ the Poisson bracket is written as

$$\{F, G\} = \frac{\partial F}{\partial q} \cdot \frac{\partial G}{\partial p} - \frac{\partial F}{\partial p} \cdot \frac{\partial G}{\partial q}\ .$$

Recall that for a function $F \in C^0(M)$ we denote by $||F||$ its uniform norm $\max_M |F|$ and by $\langle F \rangle$ its mean value $\int_M F\omega^n$. A Hamiltonian function $F \in C^\infty(M \times [0;1])$ is called *normalized* if $\langle F_t \rangle = 0$ for all t.

We refer to [10, 17] for further preliminaries on Hamiltonian diffeomorphisms.

1.3 Quasi-morphisms

A real-valued function μ on a group Γ is called a *homogeneous quasi-morphism* [2] if

(i) There exists $C > 0$ so that

$$|\mu(\varphi\psi) - \mu(\varphi) - \mu(\psi)| \leq C \ \text{ for all elements } \varphi, \psi \in \Gamma;$$

(ii) $\mu(\varphi^m) = m\mu(\varphi)$ for each $\varphi \in \Gamma$ and each $m \in \mathbb{Z}$.

The minimal constant C in the above inequality is called *the defect* of μ.

In this paper we will deal with homogeneous quasi-morphisms on the group $\mathcal{G}$ with the following property:

$$\int_0^1 \min_M (F_t - G_t)\, dt \leq \mu(\phi_G) - \mu(\phi_F) \leq \int_0^1 \max_M (F_t - G_t)\, dt \qquad (3)$$

for all **normalized** Hamiltonians $F, G \in C^\infty(M \times [0;1])$. We call them *stable* quasi-morphisms.

Example 1.1. The group $\mathcal{G}$ is known to carry a stable homogeneous quasi-morphism for the following list of symplectic manifolds [5, 7, 15]: complex projective spaces and Grassmannians; $\mathbb{C}P^2$ blown up at $k \leq 3$ points with a symplectic form in a rational cohomology class; strongly semi-positive (see

Section 1.5) direct products of the above-mentioned manifolds. The existence of stable homogeneous quasi-morphisms is related to the algebraic structure of the quantum homology of (M, ω). See Section 4.2 below for more discussion on the stability property.

Given a stable homogeneous quasi-morphism μ, define a functional $\zeta : C^\infty(M) \to \mathbb{R}$ by

$$\zeta(F) = \int_M F\omega^n - \mu(\phi_F) \; . \tag{4}$$

Recall that the Lie algebra of the group $\mathcal{G}$ can be identified with the space $C_0^\infty(M)$ of smooth functions on M with zero mean. With this language the restriction of ζ to $C_0^\infty(M)$ is simply the pullback of quasi-morphism $-\mu$ on the group to the Lie algebra via the exponential map. One can show that ζ satisfies the axioms of a symplectic quasi-state listed in Section 1.1: Axiom (i) is obvious (since, according to our normalization, $\int_M \omega^n = 1$), axiom (ii) is a simple corollary of the stability property (3) of μ (see Section 4.1 below) and axiom (iii) follows from the fact (which is an easy exercise) that the restriction of any homogeneous quasi-morphism to an abelian subgroup is a homomorphism (see [6]).

As an immediate consequence of axiom (ii) we get that ζ is 1-Lipschitz with respect to the uniform norm and thus extends to the space of continuous functions $C^0(M)$. Furthermore, the functional

$$\Pi(F, G) = |\zeta(F + G) - \zeta(F) - \zeta(G)|$$

(see formula (1) above) is Lipschitz as well:

$$|\Pi(F, G) - \Pi(F', G')| \leq 2(||F - F'|| + ||G - G'||) \tag{5}$$

for all functions $F, G, F', G' \in C^\infty(M)$.

It is important to emphasize that in the setting of Example 1.1 above the quasi-state ζ is non-linear, that is $\Pi(F, G) > 0$ for some $F, G \in C^\infty(M)$.

Example 1.2. Even for the 2-sphere with the standard area form of total area 1 the explicit calculation of $\mu(f)$ for Hamiltonian diffeomorphisms f generated by a generic time-dependent Hamiltonian is a transcendentally difficult problem. However the corresponding quasi-state ζ has an easy-to-handle combinatorial interpretation, see [5]: Since ζ is Lipschitz in the

uniform norm, it suffices to define its value on the dense subset consisting of Morse functions F on the sphere with distinct critical values. Look at the set of connected components of the level lines of F. One can show that there exists unique component, say, γ_F with the following property: the area of every connected component of the complement $\mathbb{S}^2 \setminus \gamma_F$ is $\leq \frac{1}{2}$. Then $\zeta(F)$ is simply the value $F(\gamma_F)$. Moreover, if $u : \mathbb{R} \to \mathbb{R}$ is any smooth function, $\zeta(u \circ F) = u(\zeta(F))$.

Example 1.3. Think of the 2-sphere $\mathbb{S}^2$ as of the Euclidean unit sphere in $\mathbb{R}^3(x, y, z)$ with the center at zero. Let ω be the induced area form on $\mathbb{S}^2$ divided by 4π. We claim that $\Pi(x^2, y^2) = 1$ and hence ζ is non-linear. Indeed, apply the explicit formula for ζ presented in the example above: Note that for $F(x, y, z) = x$ the component γ_F is the equator $\{x = 0\}$, and so $\zeta(x^2) = 0$. Similarly $\zeta(y^2) = \zeta(z^2) = 0$. On the other hand

$$\zeta(x^2 + y^2) = \zeta(1 - z^2) \stackrel{*}{=} 1 - \zeta(z^2) = 1 ,$$

where equality $(*)$ is valid in view of axioms (i),(iii) of a quasi-state. Summing up,

$$\Pi(x^2, y^2) = |\zeta(x^2 + y^2) - \zeta(x^2) - \zeta(y^2)| = 1 ,$$

which proves the claim.

1.4 A lower bound for the Poisson bracket

Let (M, ω) be a closed symplectic manifold. Assume that the group $\mathcal{G}$ admits a stable homogeneous quasi-morphism with defect C. Let ζ be the corresponding quasi-state and let Π be the functional defined by (1).

Theorem 1.4.

$$\Pi(F, G) \leq \sqrt{2C \cdot ||\{F, G\}||}$$

for all $F, G \in C^\infty(M)$.

The proof is given in Section 2.1.

Let us describe an application of this result to C^0-robust lower bounds on the Poisson bracket. We start our discussion with the following result from [4]:

$$F_n \xrightarrow{C^0} F,\ G_n \xrightarrow{C^0} G,\ \{F_n, G_n\} \xrightarrow{C^0} 0\ \Rightarrow \{F, G\} \equiv 0 \qquad (6)$$

(all functions are assumed to be smooth). For the sake of completeness, we present a proof in Section 3 below. In particular, if $\{F, G\} \not\equiv 0$,

$$\exists\, \epsilon_0 = \epsilon_0(F,G) > 0 \ : \ \forall\, \epsilon < \epsilon_0 \ \ \exists\, \delta = \delta(\epsilon, F, G) > 0$$

so that for all smooth functions F', G' with

$$||F - F'|| + ||G - G'|| \leq \epsilon$$

we have $||\{F', G'\}|| \geq \delta$.

This gives rise to the following definitions. Given $\epsilon > 0$ consider an open C^0-neighborhood U_ϵ of (F, G) in $C^\infty(M) \times C^\infty(M)$ defined as

$$U_\epsilon := \{ \ (F', G') \in C^\infty(M) \times C^\infty(M) \ \ | \ \ ||F' - F|| + ||G' - G|| < \epsilon \ \}.$$

Set

$$\Upsilon_{F,G}(\epsilon) := \inf_{U_\epsilon} \|\{F', G'\}\|$$

and

$$\Upsilon(F,G) := \lim_{\epsilon \searrow 0} \Upsilon_{F,G}(\epsilon) = \liminf_{F', G' \xrightarrow{C^0} F, G} ||\{F', G'\}|| \ .$$

The largest ϵ_0 as above, denoted by $\epsilon_{max}(F, G)$, can be represented as follows:

$$\epsilon_{max}(F,G) = \sup \ \{ \ \epsilon \mid \Upsilon_{F,G}(\epsilon) > 0 \ \}.$$

It reflects the size of the "maximal" neighborhood U_ϵ of (F, G) which does not contain a pair of Poisson-commuting functions. Given a positive $\epsilon < \epsilon_{max}$ one can pick the corresponding $\delta(\epsilon, F, G)$ (see above) as $\Upsilon_{F,G}(\epsilon)$.

It is an interesting problem to find explicit (lower) estimates for the numbers $\Upsilon(F, G)$, $\epsilon_{max}(F, G)$ and the function $\Upsilon_{F,G} : (0, \epsilon_0) \to \mathbb{R}$ (for at least some $\epsilon_0 \in (0, \epsilon_{max})$) in terms of F and G. As we shall see in Section 3 below, the proof of (6) leads to such estimates which involve Hofer's norm of the commutator of the Hamiltonian diffeomorphisms generated by F and G (see formulae (22), (23) below). Theorem 1.4 gives us an expression of a different nature, namely in terms of $\Pi(F, G)$, provided $\Pi(F, G) \neq 0$. As a consequence, in some examples, explicit estimates on $\Upsilon(F, G)$, $\epsilon_{max}(F, G)$ and $\Upsilon_{F,G}(\epsilon)$ can be easily obtained using the machinery of symplectic quasi-states.

Corollary 1.5. *Let $F, G \in C^\infty(M)$ be two functions with $\Pi(F, G) \neq 0$. Then*

$$\|\{F', G'\}\| \geq \frac{(\Pi(F,G) - 2\|F - F'\| - 2\|G - G'\|)^2}{2C}$$

for all $F', G' \in C^\infty(M)$ *with*

$$||F - F'|| + ||G - G'|| \leq \frac{\Pi(F,G)}{2} \; .$$

In particular, $\Upsilon(F,G) \geq (\Pi(F,G))^2/2C$*,* $\epsilon_{max}(F,G) \geq \Pi(F,G)/2$ *and* $\Upsilon_{F,G}(\epsilon) \geq (\Pi(F,G) - 2\epsilon)^2/2C$ *for all* $\epsilon \in (0, \Pi(F,G)/2)$*.*

This is an immediate consequence of Theorem 1.4 and inequality (5).

The calculation of the defect C is so far an open problem even in the simplest examples. However, upper bounds $C \leq C_0$ are available. In view of Corollary 1.5, if $\Pi(F,G) \neq 0$ we can pick $\epsilon_0(F,G) = \Pi(F,G)/2$ and give the following estimates:

$$\Upsilon(F,G) \geq (\Pi(F,G))^2/2C_0, \quad \Upsilon_{F,G}(\epsilon) \geq \frac{(\Pi(F,G) - 2\epsilon)^2}{2C_0} \quad \forall\, \epsilon \in (0, \epsilon_0) \; . \quad (7)$$

Let us illustrate these inequalities in specific examples. We start with the functions $F = x^2$ and $G = y^2$ on the two-sphere $\mathbb{S}^2$ (see Examples 1.2 and 1.3 above).

Proposition 1.6. *The quasi-state* ζ *given in Example 1.2 is induced by a stable homogeneous quasi-morphism with defect* $C \leq 2$*.*

This is proved in [5] with the exception of the upper bound on the defect. We derive this bound in Section 4.3 below.

Taking into account that $\Pi(x^2, y^2) = 1$ (see Example 1.3 above) and that $C_0 = 2$ we get that

$$\epsilon_{max}(x^2, y^2) \geq 1/2 \;\text{ and }\; \Upsilon_{x^2,y^2}(\epsilon) \geq (1 - 2\epsilon)^2/4 \;\; \forall \epsilon \in (0, 1/2) \; . \qquad (8)$$

We refer to Section 5 for further discussion of inequality (8).

Let us now outline what happens in a higher dimensional example. Consider the product of two spheres $M := \mathbb{S}^2 \times \mathbb{S}^2$ equipped with the split symplectic form $\omega \oplus \omega$. Let μ be the stable homogeneous *Calabi* quasi-morphism on $\mathcal{G}$ defined in [5] (see Example 1.1 above). The proof of the fact that μ is a quasi-morphism presented in [5] is constructive, and hence gives rise to an upper bound for the defect C of μ. In particular, unveiling the argument

presented [5, Section 3.3] in the case of $\mathbb{S}^2 \times \mathbb{S}^2$ one gets the upper bound $C \leq 6$ (the details are somewhat technical and will be omitted). Denote by ζ the quasi-state associated to μ by formula (4). Put $F = x_1^2$ and $G = y_1^2$ where (x_1, y_1, z_1) are the Euclidean coordinate functions on the first factor of M. It is an immediate consequence of [3, 6] that $\Pi(F, G) = 1$. Thus Corollary 1.5 yields

$$\epsilon_{max}(x_1^2, y_1^2) \geq 1/2 \text{ and } \Upsilon_{x_1^2, y_1^2}(\epsilon) \geq (1 - 2\epsilon)^2/12 \ \forall \epsilon \in (0, 1/2) .$$

1.5 A restriction on partitions of unity

A subset $U \subset M$ is called *displaceable* if there exists a Hamiltonian diffeomorphism ψ of M such that $\psi(U) \cap \text{Closure}(U) = \emptyset$.

In the situation of Example 1.1 the quasi-state ζ has the following additional *vanishing property*: $\zeta(F) = 0$ provided F has a displaceable support[1].

In this section we present a simple consequence of Theorem 1.4 which, in particular, provides a restriction on partitions of unity on M subordinate to coverings of M by sufficiently small sets.

Theorem 1.7. *There exists a constant $K > 0$, which depends only on the symplectic manifold (M, ω), with the following property: Given any N functions $\rho_1, \dots, \rho_N$ on M with displaceable supports so that $\sum_{i=1}^N \rho_i \geq 1$, the following inequality holds:*

$$\max_{i,j} ||\{\rho_i, \rho_j\}|| \geq \frac{K}{N^3} . \tag{9}$$

Proof. Denote by $a := \max_{i,j} ||\{\rho_i, \rho_j\}||$ the number in the left-hand side of (9). For an integer $k \in [1; N]$ put $r_k = \rho_1 + \ldots + \rho_k$. Note that $\zeta(\rho_k) = 0$ by the vanishing property. Thus Theorem 1.4 yields

$$|\zeta(r_k) - \zeta(r_{k-1})| \leq \sqrt{2C||\{r_{k-1}, \rho_k\}||} \leq \sqrt{2Ca}\sqrt{k-1} .$$

Sum up these inequalities for $k = 2, \dots, N$. Note that $\zeta(r_N) \geq \zeta(1) = 1$ in view of monotonicity axiom (ii) of ζ. Furthermore $\zeta(r_1) = \zeta(\rho_1) = 0$. Hence we get that

$$1 \leq \sqrt{2Ca} \sum_{k=2}^N \sqrt{k-1} \leq \text{const} \cdot a^{\frac{1}{2}} N^{\frac{3}{2}} ,$$

[1]In [6], the vanishing property, together with the invariance under $Symp_0(M, \omega)$, was included into the definition of a symplectic quasi-state. Today we believe that they should be considered as additional properties rather than axioms.

which yields (9). $\square$

A GENERALIZATION: Interestingly enough, a slightly weaker version of Theorem 1.7 holds true for a much more general class of closed symplectic manifolds (M, ω) than we considered before. For technical reasons we assume that M is **rational**, i.e. the image of $\pi_2(M)$ under the cohomology class of ω is a discrete subgroup of $\mathbb{R}$. Furthermore, we assume that M is **strongly semi-positive**, that is

$$2 - n \leq c_1(A) < 0 \Longrightarrow \omega(A) \leq 0, \text{ for any } A \in \pi_2(M), \tag{10}$$

where c_1 stands for the 1st Chern class of (M, ω). For instance, every symplectic 4-manifold is strongly semi-positive. We believe that eventually these assumptions will be omitted.

Fix a displaceable open subset $U \subset M$. A closed subset $X \subset M$ is called *dominated by* U if there exists a Hamiltonian diffeomorphism ψ of M with $\psi(X) \subset U$.

Theorem 1.8. *There exists a constant $K > 0$ which depends only on the symplectic manifold (M, ω) and on the subset U with the following property: Given any N functions $\rho_1, \ldots, \rho_N$ on M whose supports are dominated by U so that $\sum_{i=1}^N \rho_i \geq 1$, the following inequality holds:*

$$\max_{i,j} ||\{\rho_i, \rho_j\}|| \geq \frac{K}{N^3} \; . \tag{11}$$

The proof repeats the argument above with one modification: a reference to Theorem 1.4 is replaced by its weaker version, see Section 2.2 for the details.

1.6 Simultaneous measurability in classical mechanics

Symplectic quasi-states are classical analogues of quasi-states in quantum mechanics. The latter appeared as an attempt to revise von Neumann's notion of a quantum mechanical state as a *linear* functional on the algebra of observables. A number of physicists considered the equation $\xi(A + B) = \xi(A) + \xi(B)$, where ξ is a state and A, B are observables, as lacking physical meaning unless A and B commute: indeed, non-commuting observables are not simultaneously measurable and hence the expression $\xi(A+B)$ is not well defined. This gave rise to the notion of quasi-state, a non-linear functional

which is linear on any subspace generated by a pair of commuting observables (compare with axiom (iii) of a symplectic quasi-state). We refer to [6] for a detailed historical account.

In view of this discussion, the existence of non-linear symplectic quasi-states on classical observables (that is on functions on symplectic manifolds) naturally leads us to the problem of simultaneous measurability in classical mechanics. This problem appears in physics literature (see e.g. books by Peres [16, Chapter 12-2] and Holland [9]) as a toy example motivating the theory of quantum measurements. Below we analyze simultaneous measurability in classical mechanics in the framework of a measurement procedure, called the *pointer model*. We shall show that $\Pi(F_1, F_2)$ gives a lower bound for the error of simultaneous measurement of observables F_1 and F_2.

Consider two observables $F_1, F_2 \in C^\infty(M)$. Let $\widehat{M} = M \times \mathbb{R}^4(p,q)$, $p = (p_1, p_2)$, $q = (q_1, q_2)$, be the extended phase space equipped with the symplectic form $\widehat{\omega} = \omega + dp \wedge dq$. The $\mathbb{R}^4$ factor corresponds to the measuring apparatus (the pointer), whereas q is the quantity read from it. The coupling of the apparatus to the system is carried out with the aid of the Hamiltonian function $p_1F_1(x) + p_2F_2(x)$. The Hamiltonian equations of motion with the initial conditions $q(0) = 0, p_1(0) = p_2(0) = \epsilon$ and $x(0) = y$ are as follows:

$$\begin{aligned} \dot{q}_i &= F_i,\ i = 1,2 \\ \dot{p} &= 0 \\ \dot{x} &= \epsilon\, \mathrm{sgrad}(F_1 + F_2)\ . \end{aligned}$$

Denote by g_t the Hamiltonian flow on M generated by the function $G = F_1 + F_2$. Then $x(t) = g_{\epsilon t} y$. Let $T > 0$ be the duration of the measurement. By definition, the output of the measurement procedure is a pair of functions F_i', $i = 1,2$, on M defined by the average displacement of the q_i-coordinate of the pointer:

$$F_i'(y) = \frac{1}{T}(q_i(T) - q_i(0)) = \frac{1}{T}\int_0^T F_i(x(t))dt = \frac{1}{T}\int_0^T F_i(g_{\epsilon t} y)dt\ .$$

Note that for $\epsilon = 0$ we have $F_i' = F_i$. This justifies the above procedure as a measurement of F_i and allows us to interpret the number ϵ as *an imprecision of the pointer.*

Define *the error of the measurement* as

$$\Delta(T, \epsilon, F_1, F_2) = \|F_i' - F_i\|\ .$$

Note that in our setting this quantity does not depend on $i \in \{1; 2\}$ since the sum $F_1 + F_2$ is constant along the trajectories of g_t.

Theorem 1.9. *For all $T, \epsilon > 0$ and $F_1, F_2 \in C^\infty(M)$*

$$\Delta(T, \epsilon, F_1, F_2) \geq \frac{1}{2}\Pi(F_1, F_2) - \sqrt{\frac{C}{T\epsilon}} \cdot \sqrt{\min(\, \|F_1 - \langle F_1 \rangle\|, \|F_2 - \langle F_2 \rangle\| \,)}\ .$$

In particular, if $\Pi(F_1, F_2) \neq 0$ and the pointer is not ideal, that is $\epsilon > 0$, the error of the measurement is bounded from below by $\Pi(F_1, F_2)/2$ when $T \to +\infty$ uniformly in ϵ. Theorem 1.9 is proved in Section 2.3.

ORGANIZATION OF THE PAPER: Section 2 contains proofs of Theorems 1.4, 1.8 and 1.9. Section 3 is a mock version of Section 1.4 where we revise lower bounds for the Poisson bracket in terms of Hofer's geometry. Section 4 contains proofs of auxiliary facts on quasi-morphisms and quasi-states used in the introduction. Finally, in Section 5 we discuss some open problems.

2 Proofs

2.1 Proof of Theorem 1.4:

It suffices to prove the theorem for functions F and G with zero mean. Let f_t and g_t be the flows generated by F and G. Put $H = F + G$ and $K_t = F + G \circ f_t^{-1}$. Then K_t is a normalized Hamiltonian generating the flow $f_t g_t$ and so $\phi_K = \phi_F \phi_G$. Observe that

$$\|H - K_t\| = \|G - G \circ f_t^{-1}\| = \|G \circ f_t - G\|\ .$$

Taking into account that

$$G(f_t x) - G(x) = \int_0^t \frac{d}{ds} G(f_s x) ds = -\int_0^t \{F, G\}(f_s x) ds\ ,$$

we get that

$$\|H - K_t\| = \|G \circ f_t - G\| \leq t\|\{F, G\}\|\ . \tag{12}$$

From the stability property of μ (see (3)) and inequality (12), we get that

$$|\mu(\phi_H) - \mu(\phi_K)| \leq \int_0^1 ||H - K_t||\, dt \leq \|\{F, G\}\| \cdot \int_0^1 t dt = \frac{\|\{F, G\}\|}{2}\ . \tag{13}$$

Combining this inequality with the fact that C is the defect of the quasi-morphism μ we obtain that

$$\begin{aligned}\Pi(F,G) = |\mu(\phi_{F+G}) - \mu(\phi_F) - \mu(\phi_G)| \leq \\ \leq |\mu(\phi_H) - \mu(\phi_F\phi_G)| + |\mu(\phi_F) + \mu(\phi_G) - \mu(\phi_F\phi_G)| \leq \\ \leq |\mu(\phi_H) - \mu(\phi_K)| + C \leq \frac{\|\{F,G\}\|}{2} + C.\end{aligned}$$

Finally, let us balance this inequality. Let $E > 0$ be any number. Then

$$\Pi(EF, EG) \leq E^2\frac{\|\{F,G\}\|}{2} + C\ .$$

Since Π is homogeneous, after dividing both sides by E we obtain

$$\Pi(F,G) \leq E\frac{\|\{F,G\}\|}{2} + \frac{C}{E}.$$

Choosing the optimal value $E = \sqrt{2C/||\{F,G\}||}$, we get that

$$\Pi(F,G) \leq \sqrt{2C\|\{F,G\}\|}$$

as required. □

2.2 Proof of Theorem 1.8:

Let μ and ζ be the functionals introduced in [6, Section 7]. These functionals satisfy a number of properties which are weaker than the ones of a quasi-morphism and of a quasi-state, but still these properties suffice to our purposes. Put $\Pi(F,G) = |\zeta(F+G) - \zeta(F) - \zeta(G)|$. We claim that there exists a constant $K_1 > 0$ so that

$$\Pi(F,G) \leq K_1\sqrt{||\{F,G\}||} \tag{14}$$

for all $F, G \in C^\infty(M)$ so that the support of G is dominated by U. The proof repeats *verbatim* the proof of Theorem 1.4 above. Now we repeat the proof of Theorem 1.7 given in Section 1.5 with one modification: we replace the reference to Theorem 1.4 by the reference to (14). This yields the desired result. □

2.3 Proof of Theorem 1.9:

It suffices to prove the theorem assuming that the functions F_1 and F_2 have zero mean. The proof is divided into several steps.

STEP 1: One readily checks the following scaling properties of the functional $\Delta(T, \epsilon, F_1, F_2)$:

$$\Delta(T, \epsilon, F_1, F_2) = \Delta(\epsilon T, 1, F_1, F_2) \tag{15}$$

and

$$\Delta(T, \epsilon, EF_1, EF_2) = E\Delta(ET, \epsilon, F_1, F_2) \ \ \forall E > 0 \ . \tag{16}$$

The proofs are straightforward and we omit them.

STEP 2: Put

$$F' = \frac{1}{T}\int_0^T F_1 \circ g_t \, dt \ .$$

For $s \in [0, 1]$

$$\begin{aligned} \|F' \circ g_s - F'\| &= \frac{1}{T}\left\|\int_0^T F_1 \circ g_{t+s} \, dt - \int_0^T F_1 \circ g_t \, dt\right\| \\ &= \frac{1}{T}\left\|\int_s^{T+s} F_1 \circ g_t \, dt - \int_0^T F_1 \circ g_t \, dt\right\| \\ &= \frac{1}{T}\left\|\int_T^{T+s} F_1 \circ g_t \, dt - \int_0^s F_1 \circ g_t \, dt\right\| \\ &\leq \frac{2\|F_1\|s}{T}. \end{aligned}$$

Let f_s be the flow generated by F'. Recall that $G = F_1 + F_2$ generates the flow g_s. Therefore $-G + F' \circ g_s$ generates the flow $g_s^{-1} f_s$ whose time-one map equals $\phi_G^{-1}\phi_{F'}$. Put $K = F' - G$. Since μ is stable (see formula (3)) and all our Hamiltonians are normalized we obtain

$$\begin{aligned} |\mu(\phi_G^{-1}\phi_{F'}) - \mu(\phi_K)| &\leq \int_0^1 \| - G + F' \circ g_s - (F' - G)\| ds \\ &= \int_0^1 \|F' \circ g_s - F'\| ds \leq \int_0^1 \frac{2\|F_1\|s}{T} ds \leq \frac{\|F_1\|}{T}. \end{aligned}$$

Since μ is a homogeneous quasi-morphism with defect C, we get

$$\Pi(F_1 + F_2, -F') = \left|\mu(\phi_{F'}) - \mu(\phi_G) - \mu(\phi_K)\right|$$

$$\leq \left|\mu(\phi_G^{-1}\phi_{F'}) - \mu(\phi_K)\right| + C \leq \frac{\|F_1\|}{T} + C.$$

Since Π is Lipschitz in both variables with respect to the uniform norm (see formula (5)),

$$|\Pi(F_1 + F_2, -F_1) - \Pi(F_1 + F_2, -F')| \leq 2||F_1 - F'|| = 2\Delta(T, 1, F_1, F_2) \ .$$

Observe that $\Pi(F_1+F_2, -F_1) = \Pi(F_1, F_2)$. Combining this with the previous inequality we get

$$\Delta(T, 1, F_1, F_2) \geq \frac{1}{2}\Pi(F_1, F_2) - \frac{\|F_1\|}{2T} - \frac{C}{2} \ . \tag{17}$$

STEP 3: Using (15) and (17) we get that

$$\Delta(T, \epsilon, F_1, F_2) \geq \frac{1}{2}\Pi(F_1, F_2) - \frac{||F_1||}{2T\epsilon} - \frac{C}{2} \ .$$

We shall now balance this inequality. Let $E > 0$ and $\tau = TE$. We have that

$$\Delta(T, \epsilon, EF_1, EF_2) \geq \frac{1}{2}\Pi(EF_1, EF_2) - \frac{E||F_1||}{2T\epsilon} - \frac{C}{2} \ .$$

Recall that $\Pi(EF_1, EF_2) = E\Pi(F_1, F_2)$ and, in view of (16),

$$\Delta(T, \epsilon, EF_1, EF_2) = E\Delta(ET, \epsilon, F_1, F_2) \ .$$

Substituting this into the previous inequality, dividing by E and using $\tau = TE$ we obtain that

$$\Delta(\tau, \epsilon, F_1, F_2) \geq \frac{1}{2}\Pi(F_1, F_2) - \frac{E||F_1||}{2\tau\epsilon} - \frac{C}{2E} \ .$$

This inequality is true for every $\tau > 0$ and $E > 0$. Choosing the scaling factor E in the optimal way as $E = \sqrt{C\tau\epsilon/||F_1||}$ we get that

$$\Delta(\tau, \epsilon, F_1, F_2) \geq \frac{1}{2}\Pi(F_1, F_2) - \sqrt{\frac{||F_1||C}{\tau\epsilon}} \ .$$

Since $\Delta(T, \epsilon, F_1, F_2) = \|F_1' - F_1\| = \|F_2' - F_2\|$ and $\Pi(F_1, F_2)$ are symmetric with respect to F_1, F_2, switching F_1 and F_2 in the proof above shows that

$$\Delta(\tau, \epsilon, F_1, F_2) \geq \frac{1}{2}\Pi(F_1, F_2) - \sqrt{\frac{||F_2||C}{\tau\epsilon}} \ .$$

The last two inequalities immediately yield the needed result. $\square$

3 Hofer's metric and the Poisson bracket

In this section we revise the C^0-robustness of the Poisson bracket (compare with Section 1.4 and [4]) from the viewpoint of Hofer's geometry on $Ham(M,\omega)$ (see e.g. [17] for an introduction). We work on an arbitrary closed symplectic manifold (M,ω). Denote $\mathrm{osc} F = \max_M F - \min_M F$. For a Hamiltonian F on $M \times [0;1]$ define $\psi_F \in Ham(M,\omega)$ as the time one map of the Hamiltonian flow generated by F. The group $Ham(M,\omega)$ carries a bi-invariant metric ρ called the Hofer metric which is defined as follows:

$$\rho(\psi_F, \psi_G) := \inf \int_0^1 \mathrm{osc}\, H_t \; dt,$$

where H_t is a time-dependent Hamiltonian generating $\psi_F^{-1}\psi_G$ and the infimum is taken over all such H_t. One can easily see that

$$\rho(\psi_F, \psi_G) \leq \int_0^1 \mathrm{osc}\,(F_t - G_t)\; dt \tag{18}$$

for all Hamiltonians F and G. For $a, b \in Ham(M,\omega)$ write $[a,b]$ for the commutator $aba^{-1}b^{-1}$. Denote by $\mathbb{1}$ the unit element of $Ham(M,\omega)$.

Take any $F, G \in C^\infty(M)$. Using inequality (18) and arguing as in Section 2.1 (compare with formula (13)) we get that

$$\rho(\psi_F\psi_G, \psi_{F+G}) \leq \frac{\mathrm{osc}\{F,G\}}{2}\ .$$

Switching F and G and using the bi-invariance of ρ, we conclude that

$$\rho(\mathbb{1}, [\psi_F, \psi_G]) \leq \mathrm{osc}\{F,G\}\ . \tag{19}$$

Further,

$$[a,b][\alpha,\beta]^{-1} = (a\alpha^{-1}) \cdot \alpha\Big((b\beta^{-1}) \cdot \beta\Big((a^{-1}\alpha)\cdot \alpha^{-1}(b^{-1}\beta)\alpha\Big)\beta^{-1}\Big)\alpha^{-1}\ .$$

Together with the bi-invariance of Hofer's metric and the triangle inequality this yields

$$\rho([a,b],[\alpha,\beta]) \leq 2\rho(a,\alpha) + 2\rho(b,\beta)\ . \tag{20}$$

Take any $F', G' \in C^\infty(M)$. It follows from (20) and (18) that

$$\rho([\psi_F,\psi_G],[\psi_{F'},\psi_{G'}]) \leq 2\mathrm{osc}\,(F-F') + 2\mathrm{osc}\,(G-G').$$

Applying inequality (19) to the pair of functions F', G' and using the bi-invariance of ρ we get

$$\rho(\mathbb{1}, [\psi_F, \psi_G]) \leq \operatorname{osc}\{F', G'\} + 2\operatorname{osc}(F - F') + 2\operatorname{osc}(G - G'). \tag{21}$$

Taking into account that $\operatorname{osc} H \leq 2||H||$ we conclude that

$$\|\{F', G'\}\| \geq \frac{1}{2}\rho(\mathbb{1}, [\psi_F, \psi_G]) - 2\|F - F'\| - 2\|G - G'\|. \tag{22}$$

Recalling the notations of Section 1.4 we see that

$$\epsilon_{max}(F, G) \geq \frac{1}{4}\rho(\mathbb{1}, [\psi_F, \psi_G]), \quad \Upsilon(F, G) \geq \frac{1}{2}\rho(\mathbb{1}, [\psi_F, \psi_G])$$

and moreover

$$\Upsilon_{F,G}(\epsilon) \geq \frac{1}{2}\rho(\mathbb{1}, [\psi_F, \psi_G]) - 2\epsilon \quad \forall \epsilon \in (0, \frac{1}{4}\rho(\mathbb{1}, [\psi_F, \psi_G])). \tag{23}$$

Finally, assume that $F_m \to F, G_m \to G$ and $\{F_m, G_m\} \to 0$ in the uniform norm. It follows from (22) that ψ_F and ψ_G commute. The same holds true for ψ_{tF} and ψ_{sG} with any $s, t \in \mathbb{R}$ and hence $\{F, G\} \equiv 0$. This proves implication (6) which appears in [4].

We refer to [12, 14, 18] for other recent results related to C^0-behavior of Hamiltonians.

4 Auxiliary results on quasi-morphisms and quasi-states

4.1 Proof of monotonicity axiom (ii) of ζ:

Given a stable homogeneous quasi-morphism μ on $\mathcal{G}$, let ζ be the functional defined by (4). We have to check that $\zeta(F) \leq \zeta(G)$ for $F \leq G$. This follows immediately from the inequality

$$\zeta(F) - \zeta(G) \leq \max(F - G) \tag{24}$$

for all $F, G \in C^\infty(M)$. Indeed, put $F_0 = F - \langle F \rangle$ and $G_0 = G - \langle G \rangle$ and observe that $\zeta(F) = \zeta(F_0) + \langle F \rangle$ and $\zeta(G) = \zeta(G_0) + \langle G \rangle$. Since F_0 and G_0 are normalized we can apply (3) and get that

$$\zeta(F_0) - \zeta(G_0) \leq \max(F_0 - G_0)\ .$$

Taking into account that

$$\max(F_0 - G_0) = \max(F - G) - \langle F - G \rangle \ ,$$

we get (24). $\square$

4.2 On the stability property (3)

Here we show that the homogeneous quasi-morphism μ constructed in [5] is stable. First of all let us briefly recall the construction of μ. Let QH be the even quantum homology algebra of M. Recall that $\mathcal{G}$ denotes the universal cover of the group $Ham\,(M, \omega)$.

Let

$$c : QH \times \mathcal{G} \to \mathbb{R}$$

be *the spectral invariant* introduced by Y.-G. Oh [13, 11]. Then

$$\mu(f) = -\lim_{m \to \infty} \frac{c(e, f^m)}{m} \tag{25}$$

for certain element $e \in QH$. (Recall that we normalize the symplectic form so that the symplectic volume of M is 1 and therefore μ is defined precisely as in [5]).

It is known that

$$\int_0^1 \min_M (F_t - G_t)\, dt \le c(e, \phi_F) - c(e, \phi_G) \le \int_0^1 \max_M (F_t - G_t)\, dt \tag{26}$$

for all normalized Hamiltonians $F, G \in C^\infty(M \times S^1)$, see [5, formula (2.30)].

Let F, G be two normalized Hamiltonians on $M \times [0;1]$. Without loss of generality assume that they are defined on $M \times S^1$. This can be achieved by a suitable change of time in the flows generated by F and G which alters the values of the integrals in inequality (26) in an arbitrarily small way.

Put $F_m(x,t) = mF(x, mt)$ and $G_m(x,t) = mG(x, mt)$ and note that $\phi_F^m = \phi_{F_m}$ and $\phi_G^m = \phi_{G_m}$ for all $m \in \mathbb{N}$. Applying (26) to F_m and G_m and introducing the time variable $\tau = mt$ we get that

$$m \cdot \int_0^1 \min_M (F_\tau - G_\tau)\, d\tau \le c(e, \phi_F^m) - c(e, \phi_G^m) \le m \cdot \int_0^1 \max_M (F_\tau - G_\tau)\, d\tau \ .$$

Dividing by m and passing to the limit as $m \to \infty$ we conclude with the help of (25) that

$$\int_0^1 \min_M (F_\tau - G_\tau)\, d\tau \leq \mu(\phi_G) - \mu(\phi_F) \leq \int_0^1 \max_M (F_\tau - G_\tau)\, d\tau \; .$$

□

4.3 Estimating the defect for the 2-sphere

Here we prove Proposition 1.6. We start with some preliminaries from [5]: Consider the field $k = \mathbb{C}[[s]$ of Laurent series in one variable (the series are possibly infinite in the negative direction but finite in the positive one). The quantum homology algebra QH of $\mathbb{S}^2$ is the algebra of polynomials with coefficients in k in the variable p modulo the ideal generated by $p^2 - s^{-1}$:

$$QH = k[p]/\{p^2 = s^{-1}\} \; .$$

The quasi-morphism μ is defined by formula (25) above, where $e = 1 \in QH$. Denote by $\mathbb{1}$ the unit element of $\mathcal{G}$. We shall need the following properties of the spectral invariant $c : QH \times \mathcal{G} \to \mathbb{R}$ which can be readily extracted from [5]:

(i) $c(ps, \mathbb{1}) = 1$;

(ii) $c(ab, fg) \leq c(a, f) + c(b, g)$;

(iii) $c(1, g) = -c(p, g^{-1})$

for all $a, b \in QH$ and $f, g \in \mathcal{G}$.

Proof of Proposition 1.6: We have to show that the defect C of μ satisfies $C \leq 2$. We claim that

$$c(1, f) + c(1, g) - 1 \leq c(1, fg) \leq c(1, f) + c(1, g) \qquad (27)$$

for all $f, g \in \mathcal{G}$. Indeed, the inequality on the right is an immediate consequence of property (ii) above. To get the inequality on the left we use (i), (ii) and (iii) and observe that

$$\begin{aligned} c(1, f) = c(1 \cdot p \cdot ps, fg \circ g^{-1} \circ \mathbb{1}) &\leq c(1, fg) + c(p, g^{-1}) + c(ps, \mathbb{1}) \\ &= c(1, fg) - c(1, g) + 1 \; . \end{aligned}$$

This proves (27).

It follows from inequality (27) that

$$mc(1,h) - (m-1) \le c(1,h^m) \le mc(1,h)$$

and hence formula (25) yields

$$c(1,h) - 1 \le -\mu(h) \le c(1,h) \tag{28}$$

for all $h \in \mathcal{G}$.

Take any $f, g \in \mathcal{G}$. Substituting consecutively $h = f$, $h = g$ and $h = fg$ into inequality (28) and using (27) we readily get that

$$-2 \le \mu(fg) - \mu(f) - \mu(g) \le 2 ,$$

which completes the proof. □

5 Discussion and open problems

In Section 1.4 we have defined the functional

$$\Upsilon(F,G) = \liminf_{F',G' \xrightarrow{C^0} F,G} ||\{F',G'\}|| .$$

The main open problem concerning this functional is as follows:

Question 5.1. *Is it true that we always have* $\Upsilon(F,G) = \|\{F,G\}\|$*?*

In a recent work by one of the authors [19] this question is answered in the positive for two-dimensional symplectic manifolds using methods of the topology of surfaces.

In the case when M is the sphere $\mathbb{S}^2$, the symplectic quasi-state ζ, and hence the functional Π, are defined in elementary terms (see Example 1.2 above). Thus the *formulations* of Theorems 1.4 and 1.9 are "soft". However our proofs of these theorems are "hard"[2]: they use in a crucial way the fact that ζ is induced by a stable homogeneous quasi-morphism on $\mathcal{G}$ which is defined by means of Floer homology. This somewhat paradoxical situation was partially resolved in [19]: For the sphere, Theorem 1.4 is proved in

[2]The terms "soft" and "hard" are understood here in the sense of Gromov [8].

[19] by "soft" methods. Moreover, an analogous theorem is proved with the sphere replaced by an arbitrary closed symplectic surface Σ, and with the quasi-state of Theorem 1.4 replaced by an arbitrary simple quasi-state (that is a quasi-state which is *multiplicative* on each singly generated closed subalgebra of $C(\Sigma)$). It is still unclear to us whether Theorem 1.9 on the error of a simultaneous measurement of non-commuting observables admits a "soft" proof in the case of $\mathbb{S}^2$.

At the moment the authors believe that in higher dimensions, the C^0-robustness of the Poisson bracket is a "hard" phenomenon and thus the Floer-theoretical methods used above are adequate in this context.

Another interesting circle of problems is related to the sharpness of various estimates obtained in the present work. For instance, Zapolsky's result mentioned in the beginning of this section shows that inequality (8) stating that $\Upsilon_{x^2,y^2}(\epsilon) \geq (1-2\epsilon)^2/4$ is not asymptotically (as $\epsilon \to 0$) sharp: indeed one readily computes that $||\{x^2, y^2\}|| \approx 9.7 > 0.25$. It would be interesting to understand whether the approach of [19] gives rise to sharp lower bounds for $\Upsilon_{x^2,y^2}(\epsilon)$ on surfaces.

Question 5.2. *What is the sharp value of the constant C in Theorem 1.4 stating that $\Pi(F,G) \leq \sqrt{2C \cdot ||\{F,G\}||}$?*

The answer is so far unknown even for the case of the quasi-state ζ from Example 1.2 on the 2-sphere. In this case Floer-theoretical Proposition 1.6 above yields $C \leq 2$, while the topological argument from [19] improves this to $C \leq 1/2$.

Question 5.3. *Can one improve (asymptotically in N as $N \to \infty$) the bound*

$$\max_{i,j} ||\{\rho_i, \rho_j\}|| \geq const/N^3$$

for the partitions of unity given by Theorem 1.7?

We claim that the asymptotic behavior of the right hand side cannot be made better than $\sim N^{-2}$. Indeed, fix any partition of unity $\rho'_1, ..., \rho'_d$ subordinate to a covering by displaceable subsets. For any $m \in \mathbb{N}$ consider $N = md$ functions ρ_i where $\rho_i = m^{-1} \cdot \rho'_j$ for $i = j \pmod d$. In other words, we take each function ρ'_j/m with multiplicity m. Of course, we again get a partition of unity subordinate to the covering by displaceable subsets. When $m \to \infty$,

the left hand side of the inequality above is of the order $\sim m^{-2} = d^2 N^{-2}$, and the claim follows. At the moment, we do not know an answer to Question 5.3 even in dimension two.

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Michael Entov
Department of Mathematics
Technion - Israel Institute of Technology
Haifa 32000, Israel
entov@math.technion.ac.il

Leonid Polterovich
School of Mathematical Sciences
Tel Aviv University
Tel Aviv 69978, Israel
polterov@post.tau.ac.il

Frol Zapolsky
School of Mathematical Sciences
Tel Aviv University
Tel Aviv 69978, Israel
zapolsky@post.tau.ac.il